Math 3235 Probability Theory

1/31/23

Sec 2.3. Function of discrete r.v.

if X is bernoulli with par p.

and a and b are real

numbers

$$Y = aX + b$$

Y Takes Two values $\{b, a+b\}$

$p_Y(b) = \mathbb{P}(Y = b) = (1-p)$

$p_Y(a+b) = p$

For any r.v. X I can

consider

$$Y = aX + b$$

Assume That p_X is p.m.f. of X.

$$P_X(x) = \mathbb{P}(X = x)$$

$y \in Im(Y)$ iff $\exists\, x \in Im(X)$ such that $y = ax + b$

If $y \in Im(y)$

$$\mathbb{P}(Y = y) = \mathbb{P}(aX + b = y) = \mathbb{P}\left(X = \frac{y-b}{a}\right)$$

If $Y = aX + b$

$$\mathbb{P}(Y = y) = \mathbb{P}\left(X = \frac{y-b}{a}\right)$$

$Y = X^2$

$$\mathbb{P}(Y = y) = \mathbb{P}(X = \sqrt{y}) + \mathbb{P}(X = -\sqrt{y}).$$

for $y \geq 0$.

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if f is a function $\mathbb{R} \to \mathbb{R}$ and X is a discrete r.v. with p.m.f. $p_X(x)$ Then

$$p_Y(y) = \mathbb{P}(Y=y) = \sum_{x \in f^{-1}(y)} p_X(x) = \sum_{x \mid f(x)=y} \mathbb{P}(X=x)$$

Suppose X is Binomial with par N $\frac{1}{2}$

$$Y = X - \frac{N}{2}$$

N is even.

$$\mathbb{P}(Y = y) = \binom{N}{\frac{N}{2} + y} 2^{-N}$$

$$y \in \left(-\frac{N}{2}, \frac{N}{2} \right).$$

If $X = \sum_{i=1}^{N} Z_i$ with Z_i Bernoulli.

$$Y = \sum_{i=1}^{N} \left(Z_i - \frac{1}{2} \right)$$

$Z_i - \frac{1}{2}$ Takes The Two values $-\frac{1}{2}$ and $\frac{1}{2}$ with prob 0.5.

Expected values

$$\mathbb{E}(X) = \sum_{x \in \mathrm{Im}(X)} x \, \mathbb{P}(X = x) = \sum_{x} x \, p_X(x)$$

Experiment whose outcome is described by a r.v. X.

I repeat the experiment N times (N large) and I get

$x_1, x_2, \dots \quad x_N$

$$\bar{x} = \frac{1}{N} \sum_i x_i \qquad \bar{x} \simeq \mathbb{E}(X)$$

N large.

Examples:

1) X is bernoulli

$$\mathbb{E}(X) = 1\,p(1) + 0\,p(0) = p$$

2) X binomial (N, p)

$$\mathbb{E}(X) = Np$$

$$\mathbb{E}(X) = \sum_{x=\cancel{0},1}^{N} x \binom{N}{x} p^x (1-p)^{N-x}$$

$$x \binom{N}{x} = \frac{N!}{(x-1)!(N-x)!} =$$

$$N \frac{(N-1)!}{(x-1)!\,(N-x)!} = N \binom{N-1}{x-1}$$

$$\mathbb{E}(X) = \sum_{x=1}^{N} N \binom{N-1}{x-1} p^{x} (1-p)^{N-x} =$$

$$= Np \sum_{x=1}^{N} \binom{N-1}{x-1} p^{x-1} (1-p)^{N-x}$$

$y = x-1$

$$= Np \sum_{y=0}^{N-1} \binom{N-1}{y} p^{y} (1-p)^{N-1-y}$$

$$= Np$$

$N-1 = M$

$$\sum_{y=0}^{M} \binom{M}{y} p^{y} (1-p)^{M-y} =$$

$$(1 + (1-p))^{M} = 1$$

3) Geometric par p

$$P_X(x) = (1-p)^{x-1} p$$

$$\mathbb{E}(X) = \sum_{x=1}^{\infty} x (1-p)^{x-1} p =$$

$$= p \sum_{x=0}^{\infty} x(1-p)^{x-1} =$$

$$= -p \sum_{x=0}^{\infty} \frac{d}{dp} (1-p)^{x} =$$

$$= -p \frac{d}{dp} \left(\sum_{x=0}^{\infty} (1-p)^{x} \right)$$

$$= -p \frac{d}{dp} \left(\frac{1}{p} \right) = -p \left(-\frac{1}{p^2} \right) = \frac{1}{p}$$

X r.v. $\qquad g$ is a function

$Y = g(X)$

p_X p.m.f. of X

p_Y p.m.f. of Y

$$\mathbb{E}(Y) = \sum_{y} y \, p_Y(y) =$$

$$= \sum_y y \sum_{x \mid g(x) = y} p_X(x)$$

$$= \sum_y \sum_{x \mid g(x) = y} g(x)\, p_X(x) =$$

$$\sum_x g(x)\, p_X(x)$$

Theorem. (Law of the Subconscious Statistician)

If X is a discrete r.v. and $g: \mathbb{R} \to \mathbb{R}$ Then

$$\mathbb{E}\left(g(X)\right) = \sum_x g(x)\, p_X(x)$$

X is a r.v. we define

$$Var(X) = \mathbb{E}\left((X - \mathbb{E}(X))^2\right)$$

I could Try

$$\mathbb{E}(X - \mathbb{E}(X)) =$$

$$\mathbb{E}(X) - \mathbb{E}(X) = 0$$

Not very interesting.

$$g(x) = ax + b$$

$$\mathbb{E}(aX + b) = \sum_x (ax + b) p_X(x) =$$

$$= a \sum_x x p_X(x) +$$

$$b \sum_x p_X(x)$$

$$= a \mathbb{E}(X) + b$$

$$\mathbb{E}(aX + b) = a\mathbb{E}(X) + b$$

$$\mathbb{E}(X) = \mu$$

$$\mathbb{E}(X - \mu) = \mathbb{E}(X) - \mu = 0$$

$$\mathbb{E}\left((X - \mu)^2\right) = Var(X).$$